

TAM 251 — Discussion 6

Torsion-Bar Suspension Solution

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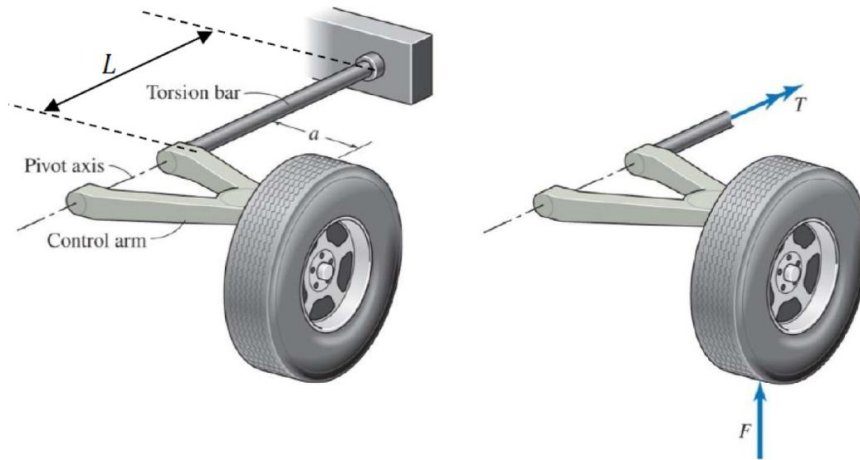
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Scope and model



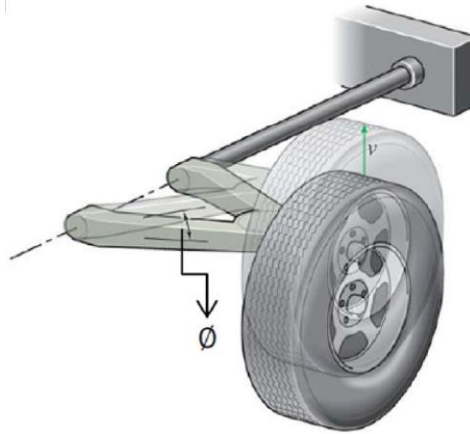
Assumptions used throughout

The control arm and chassis are rigid. The arm is initially horizontal in the plane of wheel motion, and the wheel's force acts at distance a from the shaft axis. Rotations and displacements are small. The bar is uniform, circular, homogeneous, and linearly elastic; deformation is measured from a zero-torque reference configuration.

The analysis treats the bar in **pure torsion**: ideal pivot/bearing supports carry the non-torsional force reactions. Bending, transverse shear, stress concentrations, friction, inertia, damping, and component self-weight are outside this model. Both designs use the same G , L , and a .

Geometry, free body, and torsion

1. Relate wheel displacement to shaft twist

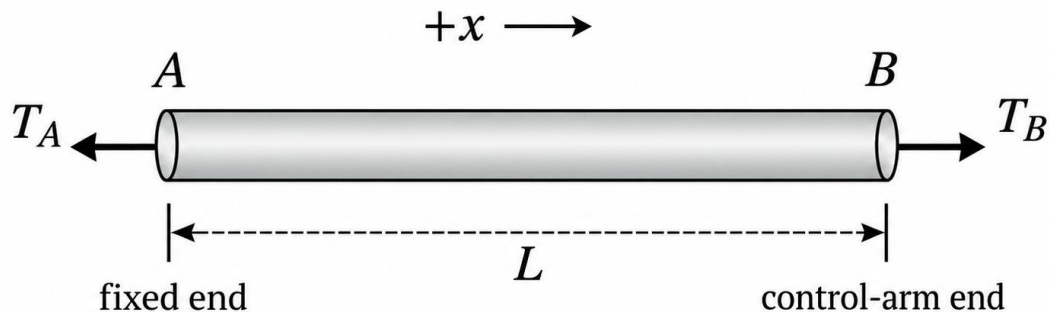


The control arm rotates through the same angle φ as the movable end of the torsion bar. For an initially horizontal rigid arm of length a , the finite-rotation geometry is $v = a \sin \varphi$. In the small-rotation model used for this worksheet,

$$v \simeq a\varphi \quad \Rightarrow \quad \boxed{\varphi \simeq \frac{v}{a}}$$

Here v is the vertical motion relative to the chassis, not an axial elongation of the bar.

2. Draw the free-body diagram of the shaft



The arrows represent torque vectors, not axial forces. The torques at A and B are equal and opposite, with magnitude Fa .

3. Relate internal torque to the applied force

Take moments about the control-arm pivot axis. Pivot forces have zero moment about that axis:

$$\sum M_{\text{pivot}} = 0: \quad Fa - T = 0 \quad \Rightarrow \quad \boxed{T = Fa}$$

4. Torque–twist relation for a solid circular bar

For uniform, linearly elastic circular-shaft torsion, $\varphi = TL/(GJ)$. For a solid diameter d

$$J_{\text{solid}} = \frac{\pi d^4}{32}, \quad \boxed{\varphi_{\text{solid}} = \frac{32TL}{\pi G d^4}}$$

Equivalently, $T = (GJ_{\text{solid}}/L)\varphi$. The quantity GJ/L is the **rotational** stiffness; it is not yet the wheel's vertical stiffness.

Solid-bar stiffness and design changes

5. Derive the vertical suspension stiffness

Combine $v \simeq a\varphi$, $T = Fa$, and $\varphi = TL/(GJ)$:

$$v = a \left(\frac{TL}{GJ} \right) = \frac{Fa^2L}{GJ}.$$

Hence, within the linearized model,

$$k_{\text{solid}} = \frac{F}{v} = \frac{GJ_{\text{solid}}}{a^2L} = \frac{\pi G d^4}{32 a^2 L}.$$

6. Compute the solid-bar stiffness

$$J_{\text{solid}} = \frac{\pi(32)^4}{32} = 102,943.708 \text{ mm}^4.$$

$$k_{\text{solid}} = \frac{80,000(102,943.708)}{(400)^2(1300)} = \boxed{39.5937 \text{ N/mm}}.$$

7. Consequences of excessive stiffness

For a specified wheel load, $v = F/k$: a stiffer suspension permits less wheel motion relative to the chassis. For a specified imposed wheel displacement, $F = kv$: a larger stiffness transmits a larger force through this idealized spring. A suspension that is too stiff for rough terrain can therefore give a harsher ride and limit the wheel's ability to accommodate surface irregularities.

Greater stiffness can be desirable when carrying larger loads while limiting suspension travel, avoiding excessive sag or bottoming out, or controlling body motion. These are qualitative design considerations, not a dynamic performance prediction. Actual behavior also depends on vehicle mass, damping, speed, and road input.

8. Three changes to reduce excessive stiffness

The following comparisons keep the applied force F and unmentioned parameters fixed. For a solid bar, $k = \pi G d^4 / (32 a^2 L)$ and $\tau_{\text{max}} = 16 F a / (\pi d^3)$.

Change	Effect on stiffness	Trade-off under the same force
Decrease diameter d	$k \propto d^4$: stiffness decreases strongly.	$\tau_{\text{max}} \propto d^{-3}$: stress increases; allowable stress must be checked.
Increase bar length L	$k \propto 1/L$: stiffness decreases.	Twist and travel increase; more installation length is needed. Nominal shear stress is unchanged.
Increase arm length a	$k \propto a^{-2}$: stiffness decreases.	$T = Fa$ and shear stress increase; the linkage needs more space and geometry changes.

Tubular bar and stiffness comparison

9. Derive the tubular-bar stiffness

For a circular annulus, the exact polar second moment of area is

$$J_{\text{tube}} = \frac{\pi}{32} (d_o^4 - d_i^4).$$

Using the same small-rotation suspension model,

$$k_{\text{tube}} = \frac{GJ_{\text{tube}}}{a^2L} = \frac{\pi G(d_o^4 - d_i^4)}{32a^2L}.$$

10. Compute the tubular-bar stiffness

$$J_{\text{tube}} = \frac{\pi(32^4 - 25^4)}{32} = 64,594.188 \text{ mm}^4.$$

$$k_{\text{tube}} = \frac{80,000(64,594.188)}{(400)^2(1300)} = \boxed{24.8439 \text{ N/mm}}.$$

11. Stiffness ratio for equal outer diameters

With the same material, bar length, and arm length, and $d_o = d$,

$$\frac{k_{\text{tube}}}{k_{\text{solid}}} = \frac{J_{\text{tube}}}{J_{\text{solid}}} = 1 - \left(\frac{d_i}{d}\right)^4.$$

Numerically, $1 - (25/32)^4 = \boxed{0.627471}$.

The tube has approximately **62.75% of the solid bar's stiffness**, or **37.25% less stiffness**, for this equal-outer-diameter comparison.

12. Which bar twists less under the same force?

The small-rotation model gives the same torque $T = Fa$ to each design. Since $J_{\text{solid}} > J_{\text{tube}}$,

$$\boxed{\varphi_{\text{solid}} < \varphi_{\text{tube}}}, \quad \frac{\varphi_{\text{tube}}}{\varphi_{\text{solid}}} = \frac{J_{\text{solid}}}{J_{\text{tube}}} = 1.59370.$$

Optional numerical check at $F = 1 \text{ kN}$ (not an additional requirement):

Quantity	Solid	Tube
Twist, φ (rad)	0.0631413	0.1006282
Twist (degrees)	3.618°	5.766°
Linearized wheel displacement, $v = a\varphi$	25.257 mm	40.251 mm

Shear stress under the 1 kN load

13. Torque, stress extrema, and radial stress plots

For either bar,

$$T = Fa = (1000)(400) = 400,000 \text{ N} \cdot \text{mm} = 400 \text{ N} \cdot \text{m}.$$

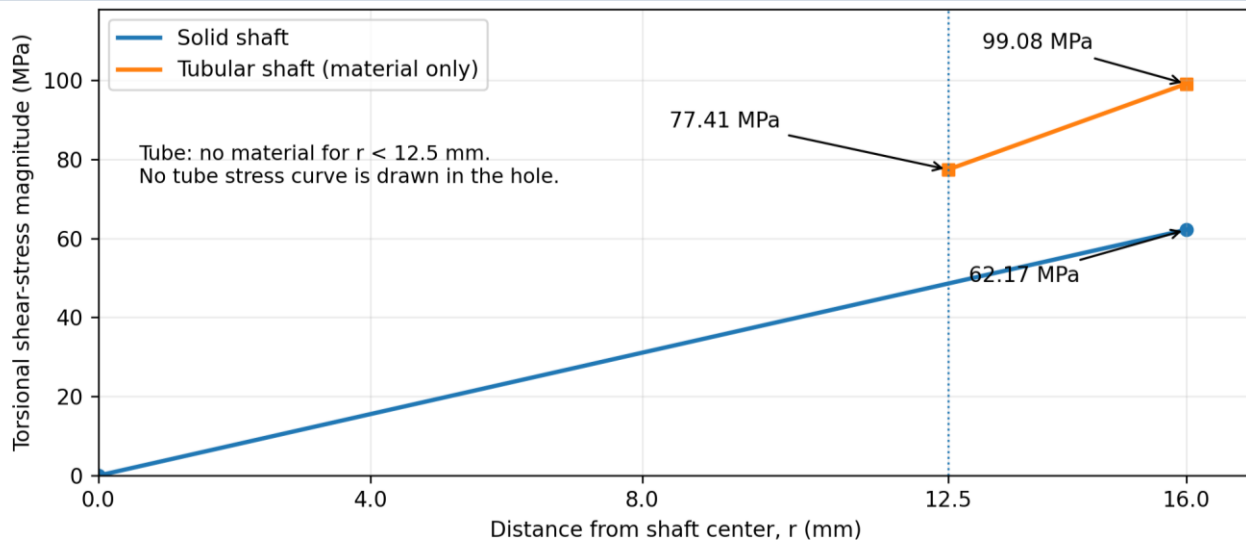
At a material point a distance r from the shaft center, the torsional shear-stress **magnitude** is $\tau(r) = Tr/J$. It varies linearly with radius within the material.

$$\tau_{\text{solid}}(r) = 3.885619 r \text{ MPa}, \quad 0 \leq r \leq 16 \text{ mm},$$

$$\tau_{\text{tube}}(r) = 6.192508 r \text{ MPa}, \quad 12.5 \leq r \leq 16 \text{ mm}.$$

In these numerical expressions, insert r in mm. The coefficients have units MPa/mm.

Cross-section	Minimum stress magnitude in the material	Maximum stress magnitude in the material
Solid	$\tau(0) = 0 \text{ MPa}$ at the center	$\tau(16) = 62.17 \text{ MPa}$ at the outer surface
Tube	$\tau(12.5) = 77.41 \text{ MPa}$ at the inner surface	$\tau(16) = 99.08 \text{ MPa}$ at the outer surface



The tube does not have material at $0 \leq r < 12.5$ mm. Do not extend its material-stress line through the hole or report zero as the tube's minimum stress in the material. The plotted minimum is at the inner wall.

These are nominal, linearly elastic torsional stresses. They are not stress-concentration values or a safety approval.

Design trade-offs and limits of the conclusion

14. Compare the designs and justify a choice

The comparison below assumes the **same outer diameter, material, bar length, arm geometry, and applied force**. It is not an equal-mass or equal-stiffness comparison.

Consideration	Solid shaft	Tubular shaft
Vertical stiffness	Higher: 39.59 N/mm	Lower: 24.84 N/mm
Twist and wheel travel at 1 kN	Smaller: 3.62° and 25.26 mm	Larger: 5.77° and 40.25 mm
Peak nominal shear stress at 1 kN	Lower: 62.17 MPa	Higher: 99.08 MPa
Material and mass	More material for the same outer size	Less material; potentially useful when reducing mass and stiffness
Practical considerations	No bore to manufacture or inspect	Wall thickness, end connections, and production method need attention

Mass comparison — additional calculation: with the same density and length, mass is proportional to cross-sectional area:

$$\frac{m_{\text{tube}}}{m_{\text{solid}}} = 1 - \left(\frac{d_i}{d}\right)^2 = 1 - \left(\frac{25}{32}\right)^2 = \boxed{0.389648}.$$

The tube therefore uses about **61.0% less mass** while retaining about **62.7% of the solid bar's stiffness**. Under the same force, its maximum nominal stress is **59.4% higher**. These quantities illustrate the trade-off; lower stiffness is not automatically inferior for a suspension.

Example defensible conclusion: I would consider the tube when reduced mass and greater suspension travel are important, provided its higher stress and larger motion satisfy the design limits. I would favor the solid bar when lower stress or reduced travel is the priority and its additional mass is acceptable. Either choice must be justified against the application's requirements; there is no uniquely preferred design from the supplied information alone.

Can either design be declared safe?

Not from Table 1 alone. The given G determines elastic stiffness, not material strength. The prompt does not supply an allowable shear stress, shear yield strength and required safety factor, a travel limit, or fatigue-loading information.

For a nominal allowable-stress check, the calculated demands are

$$\tau_{\text{allow}} \geq 62.17 \text{ MPa (solid)}, \quad \tau_{\text{allow}} \geq 99.08 \text{ MPa (tube)}.$$

If shear yield strength S_{sy} and a required factor n_{req} were provided, one could check $S_{sy}/\tau_{\text{max}} \geq n_{\text{req}}$. No strength or safety factor is invented here. Real vehicle approval would also require the load spectrum, connections, fatigue, and dynamic behavior to be assessed.